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BEAM ANALYSIS OF AXISYMMETRICAL SHELLS

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Abstract—A generalized beam theory, earlier developed by the author, is applied to a rotationally symmetrical shell structure consisting of cylindrical and conical sections. It is shown that this theory, in which the flexibility of the beam is specified by four functions, is adequate for treating the shell structure as a beam in bending in cases where the Bernoulli–Euler and Timoshenko beam theories prove to be deficient.

Flexibility functions are derived from the membrane equations of the shell structure, and modifications at the boundaries are obtained from the complete shell equations.

1. INTRODUCTION

BEAM-LIKE shell structures cannot in general be treated adequately as one-dimensional structures by the Bernoulli–Euler or the Timoshenko beam theories. However, a one-dimensional treatment with all its advantages need not be abandoned in the calculation of certain properties of beam-like shell structures provided a more general beam theory is employed. In a previous paper [1] such a beam theory was developed, and it was applied to the calculation of natural frequencies of some beam-like trusses that cannot be treated adequately by the other beam theories. In the present paper we derive the four beam functions, defining the flexibility of the beam, for an axisymmetrical thin shell composed of cylindrical and conical sections. As it is characteristic of beams with thin-walled cross-sections that the bending moment and the transverse shear force are mainly transmitted by membrane forces, we begin by deducing a set of beam functions from the membrane theory. Because of boundary effects these beam functions should be corrected in the vicinity of a junction between a conical and a cylindrical shell.

2. BEAM-LIKE STRUCTURES

The beam-like structures under consideration are straight and elastic, and symmetrical about the plane in which all loads act. An axisymmetrical thin shell is an example of such a structure provided the external loads are beam-type loads.

In Fig. 1, the positive directions are defined for the bending moment M and the transverse shear force T. The external loads are $q \,.\, \mathrm{d}x$ and $m \,.\, \mathrm{d}x$. Figure 2 shows the positive directions of the transverse deflection y, the angle of rotation of the cross-section v and the angle γ between the normal of the cross-section and the middle line.

According to Ref. [1] the constitutive equations are

$$\frac{\mathrm{d}v}{\mathrm{d}x} = a_{11}M + a_{12}T \tag{2.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - v = a_{21}M + a_{22}T \tag{2.2}$$

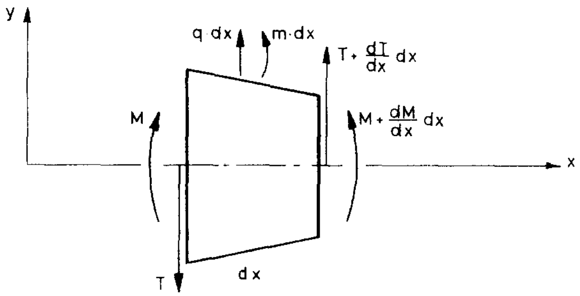


FIG. 1. Loads on an element of the beam.

where the four beam functions $a_{11}(x)$, $a_{12}(x)$, $a_{21}(x)$ and $a_{22}(x)$ determine the flexibility of the beam. The equations of equilibrium for the beam are

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{a_{21}}{D}\frac{\mathrm{d}v}{\mathrm{d}x}-\frac{a_{11}}{D}\left(\frac{\mathrm{d}y}{\mathrm{d}x}-v\right)\right\} = q \tag{2.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{-\frac{a_{22}}{D}\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{a_{12}}{D}\left(\frac{\mathrm{d}y}{\mathrm{d}x}-v\right)\right\}+\left\{\frac{a_{21}}{D}\frac{\mathrm{d}v}{\mathrm{d}x}-\frac{a_{11}}{D}\left(\frac{\mathrm{d}y}{\mathrm{d}x}-v\right)\right\} = m \tag{2.4}$$

where the function $D(x)$ is defined as

$$D(x) = a_{11}(x)a_{22}(x)-a_{12}(x)a_{21}(x). \tag{2.5}$$

Different sets of boundary conditions for the beam are mentioned in Ref. [1]. It can be shown that Maxwell's theorem results in the condition

$$a_{12} \equiv a_{21}. \tag{2.6}$$

Furthermore, the conditions

$$a_{11} > 0, \qquad a_{22} > 0, \qquad a_{11}a_{22}-a_{12}a_{21} > 0 \tag{2.7}$$

must be satisfied, as the strain energy has to be positive.

3. BEAM FUNCTIONS DERIVED FROM MEMBRANE THEORY

In this section we shall derive a set of beam functions for a conical shell, assuming that the membrane theory is valid.

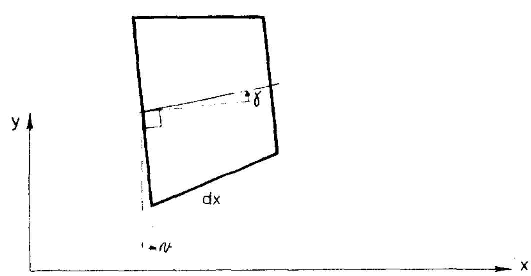


FIG. 2. Deformations of an element of the beam.

As shown in Fig. 3, we choose the coordinate system so that a point (u^1, u^2) on the middle surface of the conical shell has the cartesian coordinates

$$x^i(u^\alpha) = (u^1 \cos\psi, u^1 \sin\psi \sin u^2, u^1 \sin\psi \cos u^2). \tag{3.1}$$

The unit normal to the surface is X^i.

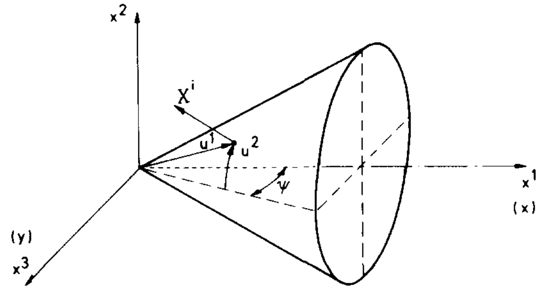


FIG. 3. Conical surface. For comparison with Figs. 1 and 2, the x–y-coordinates are shown in parentheses.

Shell theory

In the following we are going to use a set of shell equations given by Niordson [2]. These equations are acceptable in the sense of Koiter [3].

The displacements of a point on the middle surface are v^α in the directions of the surface base vectors and w in the direction of the surface normal. The deformations of the middle surface can be expressed by the membrane strain tensor

$$E_{\alpha\beta} = \tfrac{1}{2}(D_\alpha v_\beta + D_\beta v_\alpha) - d_{\alpha\beta} w \tag{3.2}$$

and the bending strain tensor

$$K_{\alpha\beta} = D_\alpha D_\beta w + d_{\alpha\gamma} D_\beta v^\gamma + d_{\beta\gamma} D_\alpha v^\gamma + v^\gamma D_\beta d_{\gamma\alpha} - d_{\beta\gamma} d^\gamma_\alpha w \tag{3.3}$$

where $d_{\alpha\beta}$ is the curvature tensor of the undeformed middle surface and D_α denotes covariant differentiation.

The external loads per unit area of the middle surface are F^α and p, acting in the directions of the surface base vectors and the surface normal, respectively. Then the equations of equilibrium can be written in the form

$$D_\alpha N^{\alpha\beta} + 2d^\beta_\gamma D_\alpha M^{\alpha\gamma} + M^{\alpha\gamma} D_\alpha d^\beta_\gamma + F^\beta = 0 \tag{3.4}$$

$$D_\alpha D_\beta M^{\alpha\beta} - d_{\alpha\beta} d^\beta_\gamma M^{\alpha\gamma} - d_{\alpha\beta} N^{\alpha\beta} - p = 0 \tag{3.5}$$

where $N^{\alpha\beta}$ is the symmetric membrane stress tensor and $M^{\alpha\beta}$ is the symmetric moment tensor.

We assume that the constitutive equations for the shell are

$$N^{\alpha\beta} = \frac{Eh}{1-\nu^2}\{(1-\nu)E^{\alpha\beta} + \nu g^{\alpha\beta} E^\gamma_\gamma\} \tag{3.6}$$

$$M^{\alpha\beta} = \frac{Eh^3}{12(1-\nu^2)}\{(1-\nu)K^{\alpha\beta} + \nu g^{\alpha\beta} K^\gamma_\gamma\} \tag{3.7}$$

where $g^{\alpha\beta}$ is the metric tensor of the undeformed middle surface, E denotes Young's modulus, ν is Poisson's ratio and h is the shell thickness.

Resultant moment and forces on a cross-section

We consider a cross-section perpendicular to the symmetry axis of a conical shell (Fig. 4). The unit normal vector to the edge is denoted n^α, and the unit tangent vector is t^α. The parameter ξ measures length along the edge of the shell.

It can be shown that the virtual work of the resultant forces and moment on the edge is

$$\delta A = \oint (T^\alpha \delta v_\alpha + Q_K \delta w + M_K \delta\theta)\, d\xi \tag{3.8}$$

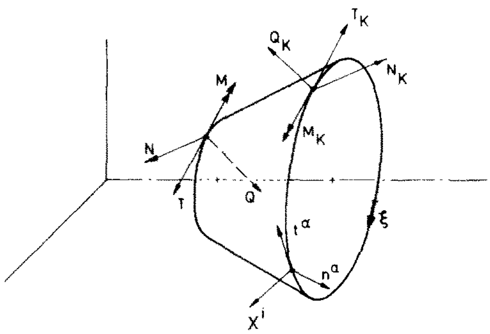


FIG. 4. Resultant moment and forces on the edge of a conical shell.

where the effective boundary membrane force per unit length is

$$T^\alpha = (N^{\alpha\beta} + d^\alpha_\gamma M^{\beta\gamma} + d^\alpha_\gamma t_\sigma t^\gamma M^{\beta\sigma}) n_\beta \tag{3.9}$$

the effective transverse force per unit length is

$$Q_K = -(D_\alpha M^{\alpha\beta}) n_\beta - \partial(M^{\alpha\beta} n_\alpha t_\beta)/\partial\xi \tag{3.10}$$

the bending moment per unit length is

$$M_K = M^{\alpha\beta} n_\alpha n_\beta \tag{3.11}$$

and θ denotes the rotation of the tangent vector. Index K refers to the conical shell, while index C will later refer to a cylindrical shell. The normal component and the tangential component of the membrane force are $N_K = T^\alpha n_\alpha$ and $T_K = T^\alpha t_\alpha$, respectively.

Fourier expansions of displacements

We now introduce the physical displacements $U = v^1$, $V = u^1 \sin\psi\, v^2$ and $W = w$, and the traditional nomenclature $(s, \phi) = (u^1, u^2)$ for the coordinates on the conical surface. Using a standard Fourier expansion of the displacement functions for an axisymmetrical shell, we assume the solution

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \sum_{j=0}^{J} \begin{Bmatrix} U_j(s)\cos(j\phi) \\ V_j(s)\sin(j\phi) \\ W_j(s)\cos(j\phi) \end{Bmatrix}. \tag{3.12}$$

Then by application of equations (3.2), (3.3) and (3.6), (3.7), we find that the resultant forces and moment per unit length of the edge can be written in the form

$$\begin{Bmatrix} N_K \\ T_K \\ Q_K \\ M_K \end{Bmatrix} = \sum_{j=0}^{\infty} \begin{Bmatrix} N_{Kj}(s)\cos(j\phi) \\ T_{Kj}(s)\sin(j\phi) \\ Q_{Kj}(s)\cos(j\phi) \\ M_{Kj}(s)\cos(j\phi) \end{Bmatrix} \tag{3.13}$$

and if we make a similar Fourier expansion of the external loads F^α and p, we find that the equations of equilibrium decouple into an infinite number of systems of three ordinary differential equations for the s-dependent functions U_j, V_j and W_j. These are given in Section 3.

As we wish to consider the shell as a beam, we shall now express the bending moment M and the transverse shear force T for the beam (Fig. 1) in terms of the resultant forces and moment on a cross-section of the shell (Fig. 4). As indicated in Fig. 3, we choose the beam coordinates so that $y = x^3$ and $x = x^1$. We calculate the contributions of the resultant moment and forces for any value of j in the expansions (3.13). The contributions to the bending moment for the beam are:

$$\begin{aligned} M_j &= \{(-N_{Kj}\cos\psi + Q_{Kj}\sin\psi)s^2\sin^2\psi + M_{Kj}s\sin\psi\} \, . \int_0^{2\pi} \cos\phi\cos(j\phi)\,\mathrm{d}\phi \\ &= \begin{cases} 0, & \text{for } j \neq 1 \\ \pi\{(-N_{Kj}\cos\psi + Q_{Kj}\sin\psi)s\sin\psi + M_{Kj}\}s\sin\psi, & \text{for } j = 1 \end{cases} \end{aligned} \tag{3.14}$$

and the contributions to the transverse shear force for the beam are:

$$\begin{aligned} T_j &= (N_{Kj}\sin\psi + Q_{Kj}\cos\psi)s\sin\psi \int_0^{2\pi} \cos\phi\cos(j\phi)\,\mathrm{d}\phi \\ &\quad - T_{Kj}s\sin\psi \int_0^{2\pi} \sin\phi\sin(j\phi)\,\mathrm{d}\phi \\ &= \begin{cases} 0, & \text{for } j \neq 1 \\ \pi(N_{Kj}\sin\psi + Q_{Kj}\cos\psi - T_{Kj})s\sin\psi, & \text{for } j = 1. \end{cases} \end{aligned} \tag{3.15}$$

Equations (3.14) and (3.15) show that only the terms with $j = 1$ in the expansions (3.12), (3.13) are able to transmit a bending moment and a transverse shear force. Consequently, only these terms are relevant when we wish to describe the shell by a beam theory.

Membrane theory

It is characteristic of beams with thin-walled cross-sections that the bending moment and the transverse shear force are mainly transmitted by the membrane forces. Therefore, it is natural first to determine the four beam functions from the membrane theory.

For simplicity, we consider a shell that is only loaded at the edge, so that the external load per unit area is $F^\alpha = p = 0$. As a result of equations (3.14) and (3.15), we are only interested in the terms with $j = 1$ in the expansions (3.12). Therefore in the rest of this section the index j is omitted so that here the functions N^{11}, N^{12}, N^{22}, U, V and W denote the s-dependent coefficients of $\sin\phi$ and $\cos\phi$ in expansions similar to (3.12). Now substituting the assumption $M^{\alpha\beta} \equiv 0$ of the membrane theory in equations (3.4), (3.5), we find the following equations of equilibrium for $j = 1$:

$$\frac{dN^{11}}{ds}+N^{21}+\frac{1}{s}N^{11}-s\sin^2\psi\, N^{22} = 0 \tag{3.16}$$

$$\frac{dN^{12}}{ds}+N^{22}+\frac{3}{s}N^{12} = 0 \tag{3.17}$$

$$s\sin\psi\cos\psi\, N^{22} = 0. \tag{3.18}$$

This simple system of ordinary differential equations has the solution

$$N^{11} = \frac{c_1}{s^2}+\frac{c_2}{s} \tag{3.19}$$

$$N^{12} = \frac{c_1}{s^3} \tag{3.20}$$

$$N^{22} = 0 \tag{3.21}$$

where the integration constants c_1 and c_2 can be determined from the boundary conditions. For a beam that is only loaded at the end, the bending moment and the transverse shear force can be written in the form

$$M = M_0 - T_0 x \tag{3.22}$$

$$T = T_0 \tag{3.23}$$

where M_0 and T_0 are constants, and

$$x = s\cos\psi. \tag{3.24}$$

When the membrane stress tensor is given by equations (3.19)–(3.21) and the moment tensor disappears, equations (3.14) and (3.15) can be written in the form

$$\begin{aligned} M &= -\pi N_K \cos\psi\, s^2 \sin^2\psi = -\pi\cos\psi\, s^2\sin^2\psi\, N^{11} \\ &= -\pi\cos\psi\sin^2\psi(c_1+c_2 s) = M_0 - T_0 x \end{aligned} \tag{3.25}$$

$$\begin{aligned} T &= \pi N_K s\sin^2\psi - \pi T_K s\sin\psi = \pi s\sin^2\psi\, N^{11} \\ &\quad -\pi s^2\sin^2\psi\, N^{12} = \pi\sin^2\psi\, c_2 = T_0. \end{aligned} \tag{3.26}$$

Now using equations (3.25) and (3.26), we replace the constants c_1 and c_2 in equations (3.19), (3.20) by the constants M_0 and T_0:

$$N^{11} = -\frac{M_0}{\pi\cos\psi\sin^2\psi}\frac{1}{s^2}+\frac{T_0}{\pi\sin^2\psi}\frac{1}{s} \tag{3.27}$$

$$N^{12} = -\frac{M_0}{\pi\cos\psi\sin^2\psi}\frac{1}{s^3}. \tag{3.28}$$

By application of equations (3.6) and (3.2) we can express the components of the membrane stress tensor in terms of the displacements. Substituting these expressions in equations (3.27), (3.28) and (3.21), we find the following three equations for the displacements of the shell

$$\frac{\mathrm{d}U}{\mathrm{d}s}+\frac{v}{s\sin\psi}V+\frac{v}{s}U+\frac{v\cot\psi}{s}W=\frac{1-v^2}{Eh}\left(-\frac{M_0}{\pi\cos\psi\sin^2\psi}\frac{1}{s^2}+\frac{T_0}{\pi\sin^2\psi}\frac{1}{s}\right) \tag{3.29}$$

$$-\sin\psi V+s\sin\psi\frac{\mathrm{d}V}{\mathrm{d}s}-U=\frac{1+v}{Eh}2s^2\sin^2\psi\left(\frac{-M_0}{\pi\cos\psi\sin^2\psi}\frac{1}{s^3}\right) \tag{3.30}$$

$$V+\sin\psi U+\cos\psi W+vs\sin\psi\frac{\mathrm{d}U}{\mathrm{d}s}=0. \tag{3.31}$$

The solution of this system of ordinary differential equations is

$$U=\frac{M_0}{\pi Eh\sin^2\psi}\frac{1}{s\cos\psi}+\frac{T_0}{\pi Eh\sin^2\psi}\ln s+C_1 \tag{3.32}$$

$$\begin{aligned}V=&\frac{M_0}{\pi Eh\sin\psi}\left(1+v-\frac{1}{2\sin^2\psi}\right)\frac{1}{s\cos\psi}\\&+\frac{T_0}{\pi Eh\sin^3\psi}(1+\ln s)-\frac{C_1}{\sin\psi}+C_2s\end{aligned} \tag{3.33}$$

$$\begin{aligned}W=&\frac{M_0}{\pi Eh\sin\psi\cos\psi}\left(\frac{1}{2\sin^2\psi}-2\right)\frac{1}{s\cos\psi}+C_1\cot\psi-C_2\frac{s}{\cos\psi}\\&-\frac{T_0}{\pi Eh\sin\psi\cos\psi}\left\{\left(1-\frac{1}{\sin^2\psi}\right)\ln s+v-\frac{1}{\sin^2\psi}\right\}\end{aligned} \tag{3.34}$$

where C_1 and C_2 are integration constants.

When the conical shell is considered as a beam, we must define the deflection y and the cross-sectional rotation v in terms of the displacements of the shell. Here we choose to define y and v by the relations

$$y=-V \tag{3.35}$$

$$v=-\frac{W\sin\psi+U\cos\psi}{s\sin\psi}. \tag{3.36}$$

By application of equations (3.32)–(3.34) and equations (3.22)–(3.24) we can express the beam deformations $\mathrm{d}v/\mathrm{d}x$ and $\mathrm{d}y/\mathrm{d}x-v$ in terms of the bending moment M and the transverse shear force T:

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{1+2\sin^2\psi}{\pi Eh\sin^3\psi}\frac{1}{x^3}M+\frac{2+v}{\pi Eh\sin\psi}\frac{1}{x^2}T \tag{3.37}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}-v=\frac{2+v}{\pi Eh\sin\psi}\frac{1}{x^2}M+\frac{2(1+v)}{\pi Eh\sin\psi}\frac{1}{x}T. \tag{3.38}$$

A comparison between these equations and equations (2.1) and (2.2) shows that application of the membrane theory leads to the following beam functions for a conical shell

$$a_{11}^{0}(x) = \frac{1+2\sin^2\psi}{\pi Eh\sin^3\psi}\frac{1}{x^3} \tag{3.39}$$

$$a_{12}^{0}(x) = -\frac{2+\nu}{\pi Eh\sin\psi}\frac{1}{x^2} \tag{3.40}$$

$$a_{21}^{0}(x) = -\frac{2+\nu}{\pi Eh\sin\psi}\frac{1}{x^2} \tag{3.41}$$

$$a_{22}^{0}(x) = \frac{2(1+\nu)}{\pi Eh\sin\psi}\frac{1}{x} \tag{3.42}$$

where $x = 0$ at the vertex of the conical surface.

Cylindrical shell

All equations for a conical shell converge towards the corresponding equations for a cylindrical shell if we carry out the limiting process:

$$\begin{cases} \psi \to 0, & s \to \infty \\ s\sin\psi = R, & ds \to dx. \end{cases} \tag{3.43}$$

Thus, the expressions (3.39)–(3.42) for the beam functions of a conical shell converge towards the following expressions for the beam functions of a cylindrical shell:

$$a_{11}^{0} = \frac{1}{\pi EhR^3} \tag{3.44}$$

$$a_{12}^{0} = 0 \tag{3.45}$$

$$a_{21}^{0} = 0 \tag{3.46}$$

$$a_{22}^{0} = \frac{2(1+\nu)}{\pi EhR}. \tag{3.47}$$

These expressions are already known from the Timoshenko beam theory.

4. CORRECTIONS OF THE BEAM FUNCTIONS

The derivation of the beam functions (3.39)–(3.42) is based on the assumption that the membrane theory is adequate. However, this assumption is unrealistic in the vicinity of a cross-section in which the generatrix of the shell contains a break, since the resultant transverse force Q_K and the resultant moment M_K are not negligible at this cross-section.

The beam-like structure shown in Fig. 5 is a thin shell composed of one conical and two cylindrical sections. To evaluate the predictions of the beam theory with the beam functions (3.39)–(3.42) and (3.44)–(3.47), we shall solve the problem shown in Fig. 5 by means of the bending theory of shells.

For cylindrical shells the exact shell equations have constant coefficients, and it is well-known that these equations can be solved analytically by exponential functions.

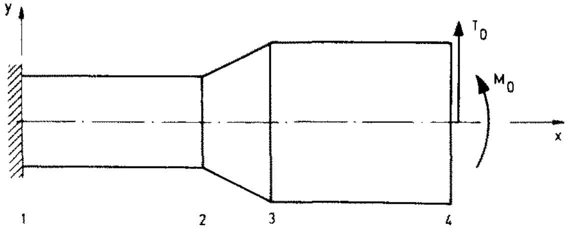


FIG. 5. An axisymmetrical shell considered as a clamped beam.

For conical shells Hoff [4] has indicated that power series solutions of a set of Donnell type equations can be obtained. However, according to Pohle [5] these series expansions are not very useful from a numerical standpoint because of poor convergence. A power series solution of a set of more exact equations has been proposed by Wan [6] who states that the convergence of the series will be slow for moderate values of j in the expansions (3.12). In the present paper we will solve the problem by a numerical analysis.

Differential equations for a conical shell

By application of the constitutive equations (3.6), (3.7) and the expressions (3.2) and (3.3) for the membrane strain tensor and the bending strain tensor, we can write the equations of equilibrium (3.4), (3.5) as a system of partial differential equations for the displacements. Substituting the Fourier expansions (3.12) in these equations, we find the following system of three ordinary differential equations for the functions U_j, V_j and W_j:

$$\begin{aligned}
&\frac{d^2U_j}{ds^2}+\frac{1}{s}\frac{dU_j}{ds}-\frac{1}{s^2}\left(j^2\frac{1-\nu}{2\sin^2\psi}+1+\frac{h^2\cot^2\psi}{12s^2}\right)U_j\\
&+j\frac{1+\nu}{2s\sin\psi}\frac{dV_j}{ds}-\frac{j}{s^2\sin\psi}\left(\frac{3-\nu}{2}+\frac{h^2\cot^2\psi}{6s^2}\right)V_j\\
&+\frac{\nu h^2\cot\psi}{12s^2}\frac{d^2W_j}{ds^2}+\frac{\cot\psi}{s}\left(\nu+\frac{h^2}{12s^2}\right)\frac{dW_j}{ds}\\
&-\frac{\cot\psi}{s^2}\left(1+\frac{h^2\cot^2\psi}{12s^2}+j^2\frac{h^2}{12s^2\sin^2\psi}\right)W_j+\frac{1-\nu^2}{Eh}F_j^1=0
\end{aligned}\tag{4.1}$$

$$\begin{aligned}
&-j\frac{1+\nu}{2s^2\sin^2\psi}\frac{dU_j}{ds}-\frac{j}{2s^3\sin^2\psi}\left(3-\nu+\frac{h^2\cot^2\psi}{3s^2}\right)U_j\\
&+\frac{1-\nu}{2s\sin\psi}\left(1+\frac{h^2\cot^2\psi}{3s^2}\right)\frac{d^2V_j}{ds^2}+\frac{1-\nu}{2s^2\sin\psi}\left(1-\frac{h^2\cot^2\psi}{3s^2}\right)\frac{dV_j}{ds}\\
&-\frac{1}{s^3\sin\psi}\left\{\frac{1-\nu}{2}\left(1-\frac{h^2\cot^2\psi}{3s^2}\right)+\frac{j^2}{\sin^2\psi}\left(1+\frac{h^2\cot^2\psi}{3s^2}\right)\right\}V_j\\
&+j\frac{h^2\cos\psi}{6s^3\sin^3\psi}\frac{d^2W_j}{ds^2}+j\frac{\nu h^2\cos\psi}{6s^4\sin^3\psi}\frac{dW_j}{ds}\\
&-j\frac{\cos\psi}{s^3\sin^3\psi}\left(1+\frac{h^2(\cot^2\psi-1)}{6s^2}+j^2\frac{h^2}{6s^2\sin^2\psi}\right)W_j+\frac{1-\nu^2}{Eh}F_j^2=0
\end{aligned}\tag{4.2}$$

$$
\begin{aligned}
&-\frac{\nu h^2 \cot\psi}{s^2}\frac{\mathrm{d}^2 U_j}{\mathrm{d}s^2}+\frac{\cot\psi}{s}\left(12\nu+\frac{(1+2\nu)h^2}{s^2}\right)\frac{\mathrm{d}U_j}{\mathrm{d}s}\\
&+\frac{\cot\psi}{s^2}\left(12+\frac{(\cot^2\psi-2+4\nu)h^2}{s^2}+j^2\frac{h^2}{s^2\sin^2\psi}\right)U_j\\
&-j\frac{2h^2\cos\psi}{s^2\sin^2\psi}\frac{\mathrm{d}^2 V_j}{\mathrm{d}s^2}+j\frac{2(2+\nu)h^2\cos\psi}{s^3\sin^2\psi}\frac{\mathrm{d}V_j}{\mathrm{d}s}\\
&+j\frac{\cos\psi}{s^2\sin^2\psi}\left(12+\frac{2h^2(\cot^2\psi-3-\nu)}{s^2}+j^2\frac{2h^2}{s^2\sin^2\psi}\right)V_j\\
&+h^2\frac{\mathrm{d}^4 W_j}{\mathrm{d}s^4}+\frac{2h^2}{s}\frac{\mathrm{d}^3 W_j}{\mathrm{d}s^3}-\frac{h^2}{s^2}\left(1+2\nu\cot^2\psi+\frac{2j^2}{\sin^2\psi}\right)\frac{\mathrm{d}^2 W_j}{\mathrm{d}s^2}\\
&+\frac{h^2}{s^3}\left(1+2\nu\cot^2\psi+\frac{2j^2}{\sin^2\psi}\right)\frac{\mathrm{d}W_j}{\mathrm{d}s}\\
&+\frac{1}{s^2}\left\{\cot^2\psi\left(12+\frac{h^2(\cot^2\psi-2-2\nu)}{s^2}\right)-j^2\frac{2h^2(2-\cot^2\psi)}{s^2\sin^2\psi}+j^4\frac{h^2}{s^2\sin^4\psi}\right\}W_j\\
&-\frac{12(1-\nu^2)}{Eh}p_j=0.
\end{aligned}
\tag{4.3}
$$

The resultant forces and moment per unit length of the edge can be expanded as shown in equation (3.13). We find the following ordinary differential expressions for the coefficient functions

$$
N_{Kj}=\frac{Eh}{1-\nu^2}\left(\frac{\mathrm{d}U_j}{\mathrm{d}s}+\frac{\nu}{s}U_j+j\frac{\nu}{s\sin\psi}V_j+\frac{\nu\cot\psi}{s}W_j\right)
\tag{4.4}
$$

$$
\begin{aligned}
T_{Kj}=\frac{Eh}{2(1+\nu)}&\left\{-\frac{j}{s\sin\psi}U_j+\left(1+\frac{h^2\cot^2\psi}{3s^2}\right)\frac{\mathrm{d}V_j}{\mathrm{d}s}\right.\\
&\left.-\left(\frac{1}{s}+\frac{h^2\cot^2\psi}{3s^3}\right)V_j+j\frac{h^2\cos\psi}{3s^2\sin^2\psi}\frac{\mathrm{d}W_j}{\mathrm{d}s}-j\frac{h^2\cos\psi}{3s^3\sin^2\psi}W_j\right\}
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
Q_{Kj}=\frac{Eh^3}{12(1-\nu^2)}&\left\{\frac{\nu\cot\psi}{s^2}\frac{\mathrm{d}U_j}{\mathrm{d}s}-\frac{(1+\nu)\cot\psi}{s^3}U_j+j\frac{2\cos\psi}{s^2\sin^2\psi}\frac{\mathrm{d}V_j}{\mathrm{d}s}\right.\\
&-j\frac{4\cos\psi}{s^3\sin^2\psi}V_j+\frac{1}{s^2}\left(1+\nu\cot^2\psi+j^2\frac{2-\nu}{\sin^2\psi}\right)\frac{\mathrm{d}W_j}{\mathrm{d}s}\\
&\left.-\frac{\mathrm{d}^3 W_j}{\mathrm{d}s^3}-\frac{1}{s}\frac{\mathrm{d}^2 W_j}{\mathrm{d}s^2}-\frac{1}{s^3}\left((1+\nu)\cot^2\psi+j^2\frac{3-\nu}{\sin^2\psi}\right)W_j\right\}
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
M_{Kj}=\frac{Eh^3}{12(1-\nu^2)}&\left\{-\frac{\nu\cot\psi}{s^2}U_j-j\frac{2\nu\cos\psi}{s^2\sin^2\psi}V_j+\frac{\mathrm{d}^2 W_j}{\mathrm{d}s^2}\right.\\
&\left.+\frac{\nu}{s}\frac{\mathrm{d}W_j}{\mathrm{d}s}-\frac{1}{s^2}\left(\nu\cot^2\psi+j^2\frac{\nu}{\sin^2\psi}\right)W_j\right\}.
\end{aligned}
\tag{4.7}
$$

Carrying out the limiting process (3.43) in equations (4.1)–(4.7), we obtain the corresponding equations for a cylindrical shell.

Boundary conditions

The form of the virtual work (3.8) indicates the types of boundary conditions that can be imposed. Thus, equilibrium conditions should be expressed in terms of the resultant forces and moment, N_K, T_K, Q_K and M_K, and geometrical conditions should be expressed in terms of the displacements, U, V and W, and the rotation, $\mathrm{d}W/\mathrm{d}s$. Index K refers to a conical shell and index C refers to a cylindrical shell.

We find that the following eight conditions must be satisfied both at cross-section 2 and at cross-section 3 of Fig. 5:

$$\begin{cases} N_{Cj} = N_{Kj}\cos\psi - Q_{Kj}\sin\psi \\ T_{Cj} = T_{Kj} \\ Q_{Cj} = N_{Kj}\sin\psi + Q_{Kj}\cos\psi \\ M_{Cj} = M_{Kj} \\ U_{Cj} = U_{Kj}\cos\psi - W_{Kj}\sin\psi \\ V_{Cj} = V_{Kj} \\ W_{Cj} = U_{Kj}\sin\psi + W_{Kj}\cos\psi \\ \mathrm{d}W_{Cj}/\mathrm{d}x = \mathrm{d}W_{Kj}/\mathrm{d}s. \end{cases} \tag{4.8}$$

The clamping of the beam at cross-section 1 can be simulated by different sets of boundary conditions. Here we choose the boundary conditions

$$\begin{cases} U_{Cj} = 0 \\ V_{Cj} = 0 \\ Q_{Cj} = 0 \\ M_{Cj} = 0 \end{cases} \tag{4.9}$$

since these conditions result in a minimal deviation from the solution of the membrane theory. Still in order to minimize the deviation from the membrane theory and also using the results (3.14) and (3.15), we choose the following boundary conditions at cross-section 4:

$$\begin{cases} N_{Cj} = \begin{cases} 0, & \text{for } j \neq 1 \\ -M_0/(\pi R^2), & \text{for } j = 1 \end{cases} \\ T_{Cj} = \begin{cases} 0, & \text{for } j \neq 1 \\ -T_0/(\pi R), & \text{for } j = 1 \end{cases} \\ Q_{Cj} = 0 \\ M_{Cj} = 0. \end{cases} \tag{4.10}$$

We note that the external load (4.10) only leads to non-trivial solutions for $i = 1$ in the Fourier expansions (3.12).

By substitution of the expressions (4.4)–(4.7) for the resultant forces and moment on the edge of a conical shell and the corresponding expressions for a cylindrical shell in the conditions (4.8)–(4.10), we can write the 24 boundary conditions as ordinary differential expressions in the displacement functions U_j, V_j and W_j.

Numerical solution

The problem shown in Fig. 5 is solved by application of a finite difference method. The differential operators are approximated by symmetrical difference operators in the equations of equilibrium (4.1)–(4.3) for a conical shell, in the corresponding equations for the cylindrical shells, and in the boundary conditions (4.8)–(4.10). Thus, the problem is reduced to the solution of a system of linear algebraic equations.

Making use of the band structure of the coefficient matrix, we solve the system of linear algebraic equations by application of Gauss-elimination.

Correction functions

By comparison, a deviation is found between predictions of the bending theory of shells and predictions of the beam theory with the membrane beam functions. This deviation is due to the inaccuracy of membrane theory in the vicinity of a junction between a conical and a cylindrical shell. However, the boundary effects in the vicinity of junctions may be taken into account in the beam theory by applying a set of corrected beam functions.

After having solved the shell equations numerically, we are able to compute those values of the beam functions that should be inserted in the beam theory in order to obtain the actual deformations of the "beam". By application of equations (3.35) and (3.36) we calculate the deflection y and the cross-sectional rotation v for two different external loads, which result in linearly independent combinations of the bending moment M and the transverse shear force T at any cross-section of the beam. Then making use of the constitutive equations (2.1) and (2.2), we are able to determine the four functions a_{11}, a_{12}, a_{21} and a_{22}. The relations between these four functions and the beam functions derived from the membrane theory define four correction functions:

$$k_{ij} = a_{ij}/a_{ij}^0 \quad \text{for } i = 1, 2, j = 1, 2. \tag{4.11}$$

According to equations (3.45) and (3.46) we have $a_{12}^0 = a_{21}^0 = 0$ for a cylindrical shell, so in these cases the correction functions have no meaning. Then we may illustrate the deviation from the membrane theory by a function $\kappa_{12} = 1 + a_{12}/C$, where C is a constant.

In Fig. 6, the four correction functions are shown for a special case of an axisymmetrical shell. The figure shows that corrections of the beam functions deduced from the membrane theory should only be made in the vicinity of cross-sections, where breaks in the generatrix occur. It also shows that the function k_{11} takes on much bigger values than the other correction functions, while the function k_{22} lies very close to unity all over the beam. As a consequence of the condition (2.6) for self-adjointness of the beam theory, the functions k_{12} and k_{21} should be identical. The reason that this requirement is not precisely satisfied at the junctions is that the cross-sections of the thin shell are very deformed here. This is not inconsistent with a linear shell theory, but it cannot be accounted for in the beam theory, where we attempt to describe the behaviour of the structure in terms of just two functions $y(x)$ and $v(x)$. A reasonable choice of the common correction function for a_{12} and a_{21} is the mean $(k_{12}+k_{21})/2$.

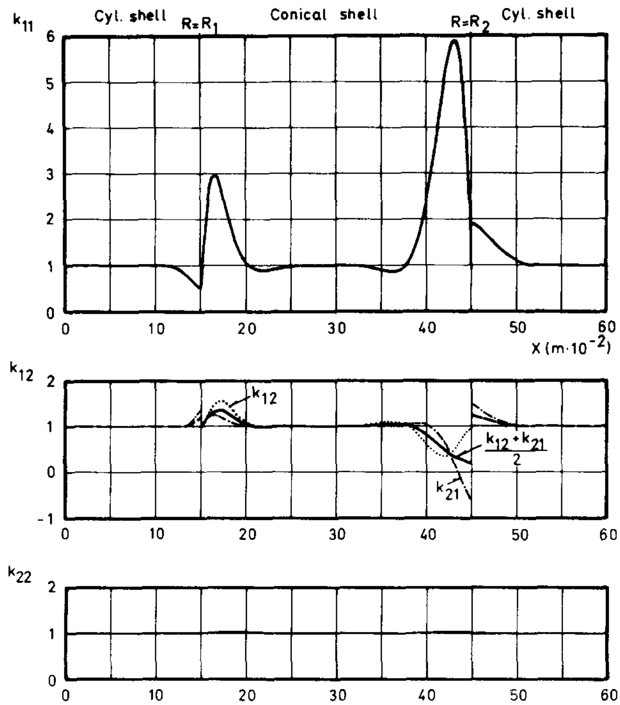


FIG. 6. Correction functions for a special case of an axisymmetrical shell, with $R_1 = 0{\cdot}100$ m, $R_2 = 0{\cdot}273$ m, $h = 0{\cdot}004$ m, $\psi = 30°$ and $\nu = 0{\cdot}30$.

The correction functions around a junction between a conical and a cylindrical shell can be non-dimensionalized in the following way

$$k_{ij} = k_{ij}\left(\frac{x}{R}, \frac{h}{R}, \psi, \nu\right) \quad \text{for } i = 1, 2 \qquad j = 1, 2 \tag{4.12}$$

where R is the radius at the junction, and x/R is the non-dimensional coordinate along the beam. The correction functions have been calculated for a constant value of Poisson's ratio $\nu = 0{\cdot}30$ and for a wide range of the parameters h/R and ψ (see Appendix A).

Until now we have only considered beam functions for conical shells where the radius increases with increasing x. We note that the beam functions a_{11} and a_{22} at a given cross-section are unaffected by a change of the positive direction of the x-axis, whereas the functions a_{12} and a_{21} change sign.

5. RESULTS

Now we are able to treat axisymmetrical shells composed of cylindrical and conical sections as beams. To find transverse deflections of such beam-like shells we solve the boundary value problem consisting of the equations of equilibrium (2.3), (2.4) and four boundary conditions. To find natural frequencies we can make use of the iterative procedure given in Ref. [1].

The beam functions to be applied in the beam theory are the functions (3.39)–(3.42) and (3.44)–(3.47) derived from the membrane theory with corrections according to equation (4.11) in the vicinity of junctions between a conical and a cylindrical shell. These beam functions are almost exactly the same as those computed by application of the bending theory of shells, and we must consequently expect that the beam solution represents the shell solution rather well. This was checked in the examples of Fig. 7.

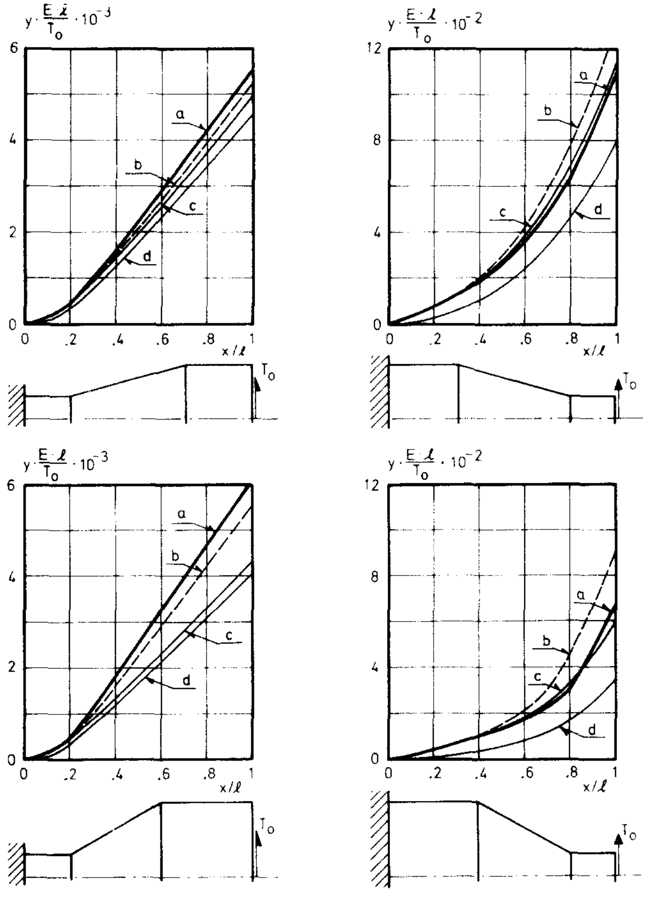


FIG. 7. Transverse deflections of beam-like shell structures. (a) Present beam theory and exact shell solution (indistinguishable from each other in the figure). (b) Beam functions a_{12} and a_{21} neglected. (c) Timoshenko beam theory. (d) Bernoulli–Euler beam theory.

Comparison with other beam theories

In the Timoshenko beam theory the beam has the constitutive equations

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{EI}M \tag{5.1}$$

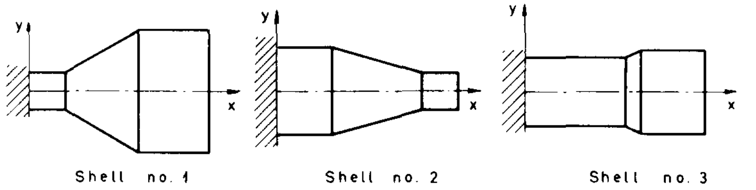
$$\frac{\mathrm{d}y}{\mathrm{d}x} - v = \frac{\mu}{GA}T \tag{5.2}$$

where the cross-sectional area is denoted A, the area moment of inertia is I, the shear modulus is G and the constant μ has the value 2 for a thin-walled circular cross-section. In the Bernoulli–Euler beam theory, the function μ/GA is neglected. We notice that the functions $1/EI$ and μ/GA are given directly by equations (3.44) and (3.47).

For the four beam-like shell structures shown in Fig. 7, the transverse deflections due to an external load are computed by application of each of the three beam theories. In the diagrams the results of the present beam theory are indistinguishable from the results of the bending theory of shells, whereas neither the Timoshenko theory nor the Bernoulli–Euler theory approximates the correct solution very well in the present four cases.

Furthermore, the transverse deflections have been calculated by application of a Timoshenko beam theory, where the functions $1/EI$ and μ/GA are replaced by the functions a_{11} and a_{22}. We note that the difference between the result of this calculation and the result of the present beam theory [curves (b) and (a) of Fig. 7] represents the influence of the terms with a_{12} and a_{21} in equations (2.1) and (2.2). The mechanism of conical shells accounted for by these two terms is actually an important part of the reason why the Timoshenko and Bernoulli–Euler beam theories are inadequate for such structures.

Furthermore, we have calculated natural frequencies ω for transverse vibrations of three beam-like shell structures. In the table of Fig. 8 the results of the three beam theories



$\Lambda = \frac{\omega^2 l^2 \rho}{E}$	Bending theory of shells	Bernoulli–Euler theory	Timoshenko theory	Present beam theory
Shell No. 1				
$\sqrt{\Lambda_1}$	0·1226	0·1512 (+23·4%)	0·1451 (+18·4%)	0·1241 (+1·3%)
$\sqrt{\Lambda_2}$	0·9108	1·676 (+84·1%)	0·8936 (−1·9%)	0·8846 (−2·9%)
$\sqrt{\Lambda_3}$	2·294	5·023 (+119·0%)	2·583 (+12·6%)	2·591 (+13·0%)
Shell No. 2				
$\sqrt{\Lambda_1}$	0·5754	0·7279 (+26·5%)	0·5649 (−1·8%)	0·5781 (+0·5%)
$\sqrt{\Lambda_2}$	1·489	2·433 (+63·4%)	1·481 (−0·6%)	1·465 (−1·6%)
$\sqrt{\Lambda_3}$	2·658	5·185 (+95·1%)	2·750 (+3·5%)	2·660 (+0·1%)
Shell No. 3				
$\sqrt{\Lambda_1}$	0·3066	0·3679 (+20·0%)	0·3178 (+3·7%)	0·3046 (−0·7%)
$\sqrt{\Lambda_2}$	1·080	2·167 (+100·7%)	1·271 (+17·7%)	1·075 (−0·5%)
$\sqrt{\Lambda_3}$	2·425	5·665 (+133·6%)	2·697 (+11·2%)	2·477 (+2·1%)

FIG. 8. Natural frequencies of shell structures. Parentheses contain relative errors in per cent.

are compared with results obtained by numerical analysis of the differential equations of the bending theory of shells. The Bernoulli–Euler beam theory gives poor approximations of the three smallest frequencies for the structures, whereas the present beam theory leads to good results in all cases except one. In the vibration analysis we use beam functions derived assuming no surface loads. However, in dynamic problems the inertial load terms enter the equations of motion of the thin shells with the effect that large deformations of the cross-sections occur at relatively high frequencies. This was pointed out by Simmonds [7] for cylindrical shells, and knowing the mode functions predicted by the bending theory of shells we may conjecture that the 13 per cent error in the third frequency of shell No. 1 is mainly due to this effect. In the important case of the first frequency of shell No. 1 the prediction of the Timoshenko beam theory is not good, and the same is true for the second and third frequencies of shell No. 3. In the other cases shown in the table the Timoshenko beam theory leads to quite good results, but normally this cannot be expected.

It should be emphasized that the beam theory presented here is useful only for shells where the geometry of the middle surface varies along the length of the shell. For cylindrical shells the beam theory reduces to the Timoshenko beam theory.

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REFERENCES

[1] V. TVERGAARD, Free vibrations of beam-like structures. *Int. J. Solids Struct.* **7**, 789–803 (1971).
[2] F. I. NIORDSON, Inledning till Skalteorin, Föreläsningar vid Danmarks tekniske Højskole. København (1968).
[3] W. T. KOITER, A Consistent First Approximation in the General Theory of Thin Elastic Shells, *Proceedings of the I.U.T.A.M. Symposium on the Theory of Thin Elastic Shells*, Amsterdam, pp. 12–33 (1960).
[4] N. J. HOFF, Thin circular conical shells under arbitrary loads. *J. appl. Mech.* **22**, 557–562 (1955).
[5] F. W. POHLE, Discussion of thin circular conical shells under arbitrary loads by N. J. HOFF. *J. appl. Mech.* **23**, 322–323 (1956).
[6] F. Y. M. WAN, On the equations of the linear theory of elastic conical shells. *Stud. appl. Math.* **49**, 69–83 (1970).
[7] J. G. SIMMONDS, Modifications of the Timoshenko beam equations necessary for thin-walled circular tubes. *Int. J. Mech. Sci.* **9**, 237–244 (1967).
[8] P. SEIDE, On the Bending of Cantilevered Thin-Walled Conical Frustums by End Loads, Ramo-Wooldridge Corporation Report No. GM-TR-284 (1957).

APPENDIX A

Correction functions around junctions

The correction functions given below are computed by numerical solution of the full shell equations. The function k_{22} equals one in all cases, and we further note that all correction functions are equal to one, when the angle ψ is zero. The function named k_{12} is actually computed as $(k_{12}+k_{21})/2$; and furthermore, as mentioned in the text below equation (4.11), this function is evaluated in a special way at cylindrical shells. The constant C mentioned there is chosen as the value of the membrane beam function a_{12}^0 for the conical shell at the junction.

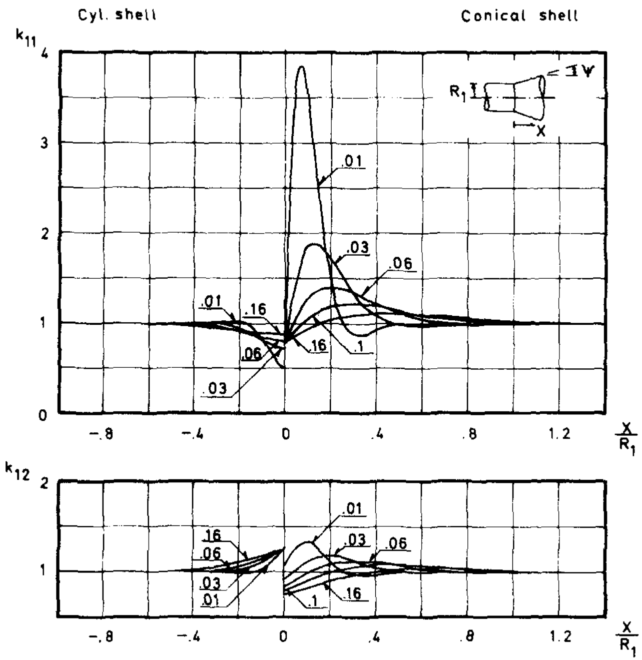

FIG. A1. Correction functions around junction between cylindrical shell and conical shell. The parameters are $\psi = 15°$, $\nu = 0{\cdot}30$ and h/R_1 as indicated on each curve.

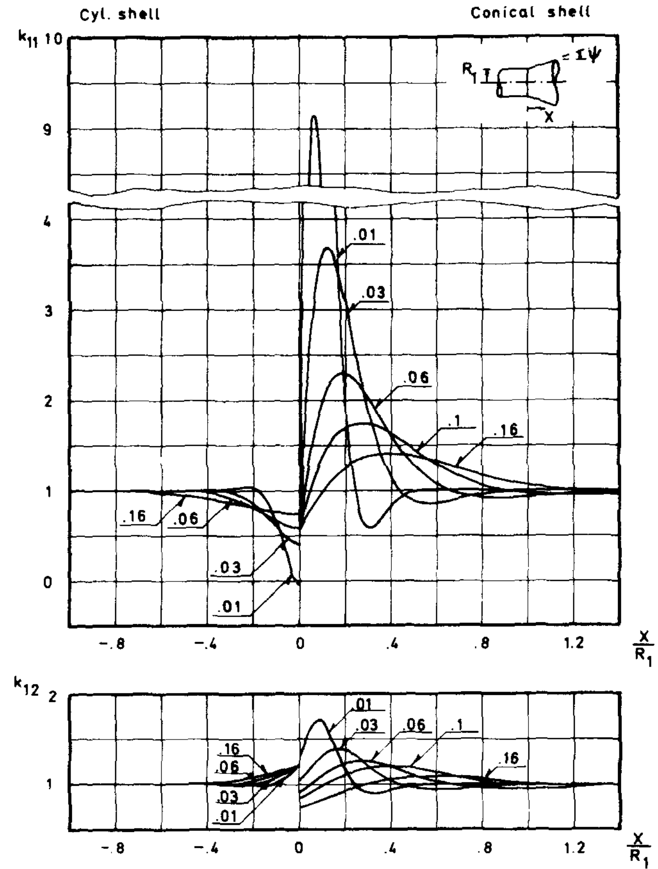

FIG. A2. Correction functions around junction between cylindrical shell and conical shell. The parameters are $\psi = 30°$, $\nu = 0{\cdot}30$ and h/R_1 as indicated on each curve.

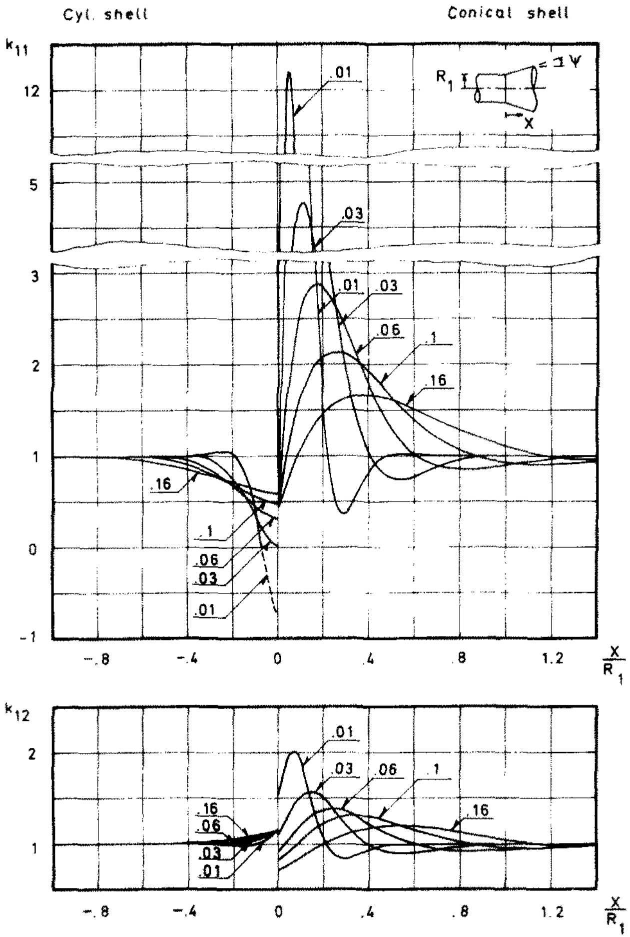

FIG. A3. Correction functions around junction between cylindrical shell and conical shell. The parameters are $\psi = 45°$, $\nu = 0{\cdot}30$ and h/R_1 as indicated on each curve.

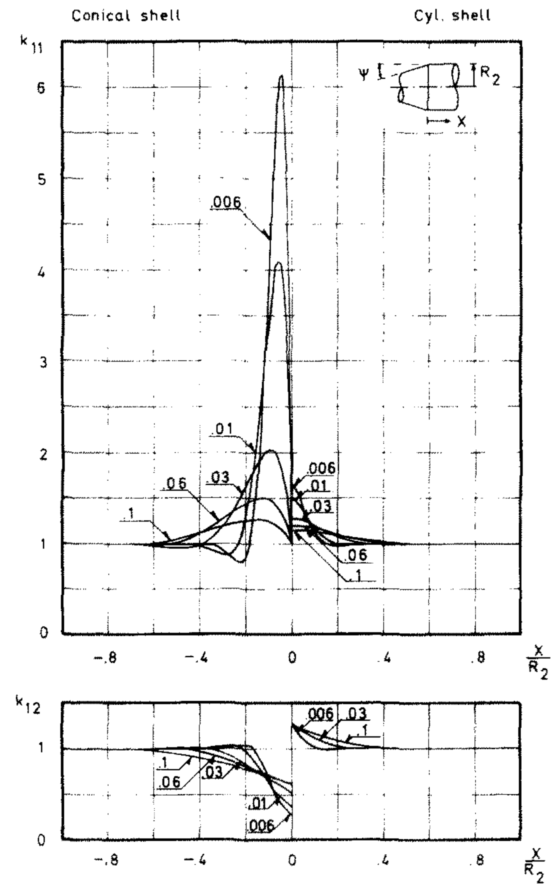

FIG. A4. Correction functions around junction between conical shell and cylindrical shell. The parameters are $\psi = 15°$, $\nu = 0{\cdot}30$ and h/R_2 as indicated on each curve.

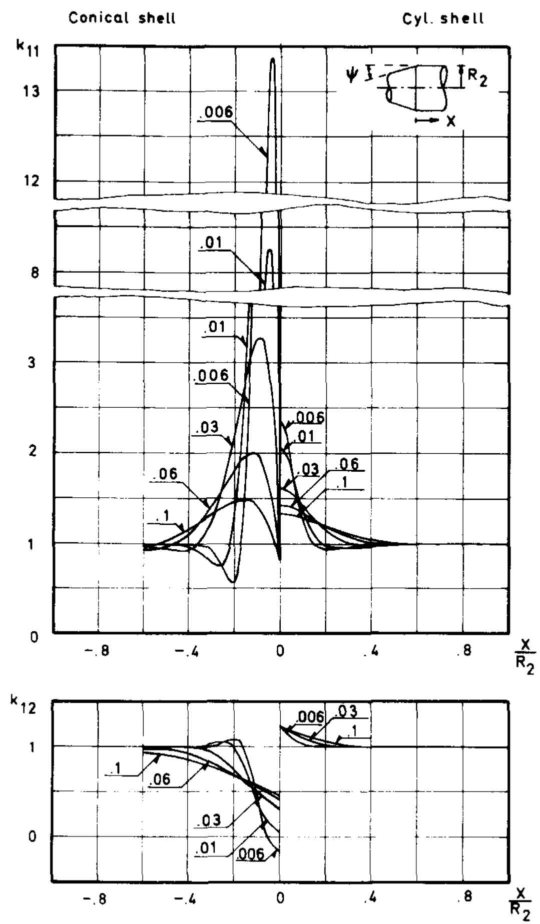

FIG. A5. Correction functions around junction between conical shell and cylindrical shell. The parameters are $\psi = 30°$, $\nu = 0{\cdot}30$ and h/R_2 as indicated on each curve.

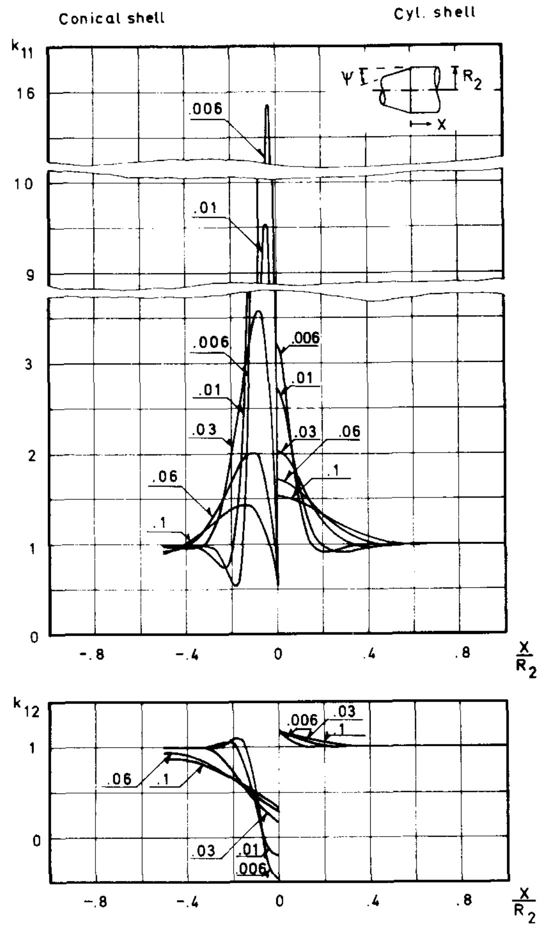

FIG. A6. Correction functions around junction between conical shell and cylindrical shell. The parameters are $\psi = 45°$, $\nu = 0{\cdot}30$ and h/R_2 as indicated on each curve.

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Абстракт—Применяется обобщенная теория балок, раньше предложенная автором, к решению симметрической оболочки вращения, состочщей из цилиндрических и конических секций. Указано, что настоящая теория, в которой гибкость балки определена четнрьмя функциями, пригодна для расчета обоиоякц в смысле изгибаемой балки, для случаев, когда теории балок Бернулли-Зйлера и Тимошенки недостаточны.

Определяются функции гибкости из уравнений безмомент-ного состояния оболоки. Получаются видоизменения на границах, изходя из полных уравнений оболочки.